Synchronization in a ring of pulsating oscillators with bidirectional couplings

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We study the dynamical behavior of an ensemble of oscillators interacting through short range bidirectional pulses. The geometry is 1D with periodic boundary conditions. Our interest is twofold. To explore the conditions required to reach fully synchronization and to invewstigate the time needed to get such state. We present both theoretical and numerical results.

The analysis of the dynamic properties of populations of pulse-coupled oscillators are the starting point of many studies devoted to understand some phenomena such as synchronization, phase locking or the emergence of spatio-temporal patterns which appear so frequently when analyzing the behavior of heart pacemaker cells, integrate and fire neurons, and other systems made of excitable units [Peskin, 1984], [Mirollo & Strogatz, 1990], [Kuramoto, 1991], [Abbott & Van Vreeswijk, 1993], [Treves, 1993].

Mean-field models or populations of just a few oscillators are the typical subject that has been considered in the scientific literature. In these simplified systems it is possible to investigate analytically the main mechanisms leading to the formation of assemblies of synchronized elements as well as other spatio-temporal structures. However, such restrictions do not allow to consider the effect of certain variables whose effect can be crucial for realistic systems. The specific topology or geometry of the system, as well as the precise connectivity between units are some typical examples which usually induce important changes in the collective behavior of these models. Unfortunately, a rigorous mathematical description of them is still missing. The majority of studies rely on simulations showing the outstanding richness that a low dimensional system of pulse-coupled oscillators may display. Some examples are self-organized criticality, chaos, quasiperiodicity, etc. [Perez et al., 1996]. In other cases, it is proven the stability of some observed behaviors [Goldsztein & Strogatz, 1995], [Diaz Guilera et al., 1997] but not the mechanisms leading to them.

A first step forward has been given very recently by [Diaz Guilera et al., 1998], hereafter called DPA. Assuming a system defined on a ring, they developed a mathematical formalism powerful enough to get analytic information not only about the mechanisms which are responsible for synchronization and formation of spatio-temporal structures, but also, as a complement, to proof under which conditions they are stable solutions of the dynamical equations. They consider one-directional interactions which allow to simplify the analysis. The study of a more general situation is desirable.

The aim of this letter is to show that such formalism is able to handle more difficult situations. In particular, we consider here a population of pulse-coupled oscillators with bidirectional couplings. This fact is important because the backwards effect of the coupling might break the coherent activity of an ensemble of oscillators previously synchronized. Small changes in local aspects of the coupling may lead to important cooperative effects.

Let us start the discussion by introducing the model and the notation used throughout the paper. We have considered a population of $(N+1)$ pacemakers distributed on a ring. This geometry is interesting to analyze certain problems related to cardiac activity. For instance, to study some types of cardiac arrhythmia characterized by an abnormally rapid heartbeat whose period is set by the time that an excitation takes to travel a circuit. This observation can be explained by modeling appropriately the circulation of a wave of excitation in a one-dimensional ring [Ito & Glass, 1992]. The study of the sinus rhythm has been also studied in systems with similar geometry and bidirectional couplings [Ikeda, 1982]. Other systems whose dynamical evolution is restricted to a limit cycle and therefore that can be described in terms of only one degree of freedom could also be tackle with the same tools.

The usual description of the state of one unit is performed in terms of one physical variable that, in general, it is voltage-like quantity. However, after a straightforward transformation [Corral et al., 1995] it is always possible to write the evolution of each oscillator in terms of a phase variable $\phi \in [0,1]$ which evolves linearly in time. In addition, the effect of pulsating-interaction between oscillators can be written down through the so called phase response curve (PRC) which measures the effective change in ϕ due to the firing process. In general, all the nonlinearities of the problem are included in this function. In this letter, we are interested in monotonic functions which are closely related to the convex character of the evolution of the voltage like variable in formal pacemak-

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ers (perfect integrators). Within this space of functions we have chosen the linear case because is allows to get analytical results without losing any of the features which characterize the synchronization process between units. Let us clarify this point. When a given unit reaches a threshold value $\phi_{th} = 1$, it fires and changes the state of its neighbors according to

$$\phi_i \geq 1 \Rightarrow \begin{cases} \phi_i \to 0 \\ \phi_{nn} \to \phi_{nn} + \varepsilon\phi_{nn} \equiv \mu\phi_{nn} \end{cases}$$

$$\forall i = 0, \ldots, N \tag{1}$$

where nn denotes the nearest neighbors, ε is the strength of the coupling and where $N+1 \equiv 0$ due to the boundary conditions. According to the aforementioned definition $\mu\phi$ is the PRC.

One of the key points of the mathematical formalism relies on a suitable transformation which allows to trace the phases of the oscillators after each firing and construct return maps of the complete cycle. For details see [Corral et al., 1995]. The transformation which intrinsically includes translations and rotations always keeps information about the element of the population which will fire immediately. This is a very appropriate way to know details about the spatio-temporal structure which is dynamically forming at every time step.

From our point of view the most direct way to understand the mechanism underlying synchronization or any other time-dependent phenomenon is to start analyzing the simplest situation with physical interest. For a population with bidirectional couplings such case is to consider only four oscillators. In DPA and for the one-directional case the same scheme was applied. However, they started with even a more simpler situation, since only three elements were matter of attention. For bidirectional interactions such case is trivial due to the symmetry of the problem since one firing affect all the neighbors and from an effective point of view the problem is mean-field which has been already solved in [Mirollo & Strogatz, 1990].

For the four pacemakers system there are six possible sequences of firings such that one member of the population fires once and only once in each cycle. If we assume that the oscillator which fires is always labeled as unit 0 and the rest of elements are ordered from this unit clockwise then these sequences are the following:

- A: 0,1,2,3
- B: 0,1,3,2
- C: 0,2,1,3
- D: 0,2,3,1
- E: 0,3,1,2
- F: 0,3,2,1

where the sequence has to be understood as the order in which oscillators reach the threshold. The dynamical process of firings can be described in terms of a set of matrices which take into account the 'jump' (distance) between those oscillators which fire consecutively [Diaz Guilera et al, 1998]. It is straightforward to check that they are

$$M_1 = \begin{pmatrix} -\mu & 1 & 0 \\ -\mu & 0 & \mu \\ -\mu & 0 & 0 \end{pmatrix} \tag{2}$$

$$M_2 = \begin{pmatrix} 0 & -1 & \mu \\ 0 & -1 & 0 \\ \mu & -1 & 0 \end{pmatrix} \tag{3}$$

$$M_3 = \begin{pmatrix} 0 & 0 & -\mu \\ \mu & 0 & -\mu \\ 0 & 1 & -\mu \end{pmatrix} \tag{4}$$

and that a all the cycles related to the different orders are constructed by combining the three previous matrices as follows

- A: 0,1,2,3 $\to T_1 \circ T_1 \circ T_1 \circ T_1$
- B: 0,1,3,2 $\to T_2 \circ T_3 \circ T_2 \circ T_1$
- C: 0,2,1,3 $\to T_1 \circ T_2 \circ T_3 \circ T_2$
- D: 0,2,3,1 $\to T_3 \circ T_2 \circ T_1 \circ T_2$
- E: 0,3,1,2 $\to T_2 \circ T_1 \circ T_2 \circ T_3$
- F: 0,3,2,1 $\to T_3 \circ T_3 \circ T_3 \circ T_3$

where T_i is defined as

$$\vec{\phi}' = T_i(\vec{\phi}) \equiv \vec{1} + M_i\vec{\phi},$$

where $\vec{\phi}'$ is a vector with N components since the zero-th component does not play any role in the description.

Before computing the fixed points of the transformation as well as the stability of the associated eigenvalues let us notice that matrices M_1, M_2 and M_3 have exactly the same structure as in the one-directional coupling case except for one column which is multiplied by μ. Therefore many properties of the new matrices can be discussed directly without the explicit calculation of them. In particular in DPA it was shown that the resultant matrices had eigenvalues with moduli larger or smaller than 1 depending on the sign of the coupling, i.e.e ε, which is rather important because it determines the stability of the fixed points and as a consequence ensures that for excitatory couplings the oscillators will synchronize their activity while for the inhibitory case complex spatio-temporal structures will be formed in the stationary state. Now, these properties do not change at all. The nature of the eigenvalues does not change

and independently of the particular position of the new fixed points the conditions for stability are the same as in the one-directional case. Even more, it is straightforward to show that the modulus of the eigenvalues that in the aforementioned case were larger than 1 are now larger and the opposite for those which are smaller than one. This fact suggests that the stability of the typical spatio-temporal structures found in DPA such as the chessboard configuration are more stable for the bidirectional case and the opposite in the other side, i.e.e, units synchronize faster.

The generalization to an arbitrary number of oscillators is a technical matter that follows the same steps discussed in DPA. For this reason the mathematical details will not be discussed here again. However, we want to stress that according to these results the physical scenario defined for a population with one-directional couplings still holds. The oscillators synchronize due to a mechanism of dimensional reduction. However, let us remark an important difference respect to the previous case. In the bidirectional case one cannot ensure that two units firing at unison in a given cycle will do it again in the next one, however it can be ensured through simple algebra that the number of synchronized oscillators cannot decrease at any time, in other words, if a couple break their mutual synchrony necessarily one of the elements of this couple will synchronize with the other neighbor. This fact shows that the term 'dimensional reduction' is more appropriate than the word 'absorption' used currently in the literature. On the other hand, in the typical inhibitory patterns the oscillators tend to be as far as possible from the nearest neighbors.

As a complement of the previous studies we have analyzed the time required to reach fully synchronization for the bidirectional case. In particular we have focused our attention in two particular situations. It is well known that for only two oscillators the maximal time needed to find both moving in synchrony is when in the initial condition the units are separated each other by $\phi = 0.5$, i.e.e when they are separated as far as possible. We wanted to check whether such phenomenon is also observed for larger populations. To do this, we have assumed that the oscillators are distributed randomly in a interval $[\phi_{min}, 1]$ where ϕ_{min} is a variable quantity. Figure 1 shows the results for different coupling strengths ε averaged over 200 samples. As we can see the randomization breaks the singular character of $\phi = 0.5$ and the time grows monotonically with the width of the interval.

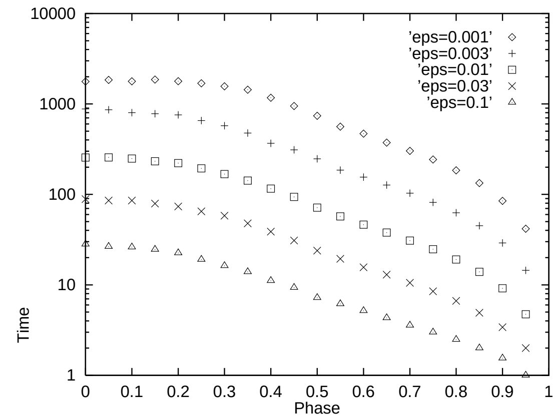

FIG. 1. Time needed to reach fully synchronization versus ϕ_{min} for several values of the coupling. The results are an average over 200 samples. The results are given for $N = 32$.

We have also studied for a fixed interval and for a fixed number of elements how depend the synchronization time with the magnitude of the coupling. In the mean-field case it was shown [Strogatz et al., 1990] that the functional dependence goes as $t \approx 1/\varepsilon$. Figure 2 shows our results for the bidirectional case. It is interesting to see that the same relationship between both variables still holds for short range couplings.

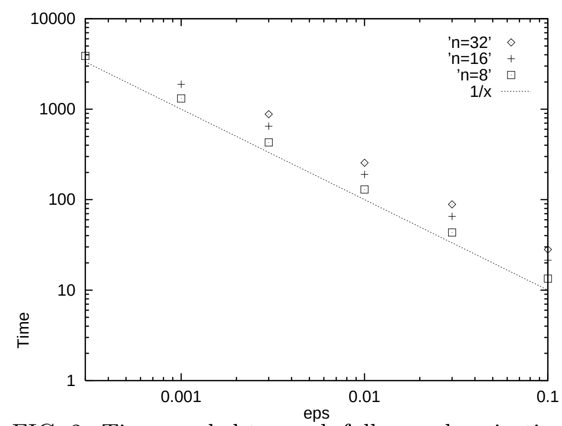

FIG. 2. Time needed to reach fully synchronization versus magnitude of the coupling for different population sizes. The results are an average over 200 samples. We have also plot the $1/x$ line as a guide for the eye.

In conclusion, we have studied under which conditions a population of pulse-coupled oscillators with bidirectional interactions display either synchronization or spatio-temporal structures. We have followed the mathematical formalism developed in DPA for a simpler case, noticing that the physical mechanisms underlying both phenomena are the same (dimensional reduction) as in the one-directional situation. We have also studied the time required to get synchronization observing the same functional dependence in the coupling found for the mean-field approach.

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